Lecture 22: Eilenberg-Zilber Theorem and Künneth formula

Eilenberg-Zilber Theorem

Definition

Let $(C_\bullet, \partial_C)$ and $(D_\bullet, \partial_D)$ be two chain complexes. We define their tensor product $C_\bullet \otimes D_\bullet$ to be the chain complex

$$(C_\bullet \otimes D_\bullet)_k := \sum_{p+q=k} C_p \otimes D_q$$

with the boundary map $\partial = \partial_{C\otimes D}$ given by

$$\partial(c_p \otimes d_q) := \partial_C(c_p) \otimes d_q + (-1)^p c_p \otimes \partial_D(d_q), \quad c_p \in C_p, d_q \in D_q.$$

This sign convention guarantees that

$$\partial^2 = 0.$$

Proposition

Assume $C_\bullet$ is chain homotopy equivalent to $C'_\bullet$. Then $C_\bullet \otimes D_\bullet$ is chain homotopy equivalent to $C'_\bullet \otimes D_\bullet$.

Proof: Assume $C_\bullet \underset{g}{\overset{f}{\rightleftarrows}} C'_\bullet$ define chain homotopy equivalence such that

$$1_{C'} - f \circ g = \partial_{C'} \circ s' + s' \circ \partial_{C'}$$
$$1_C - g \circ f = \partial_C \circ s + s \circ \partial_C$$

where

$$s : C_\bullet \to C_{\bullet+1}, \quad s' : C'_\bullet \to C'_{\bullet+1}.$$

Then our sign convention implies

$$1_{C'\otimes D} - (f \otimes 1_D) \circ (g \otimes 1_D) = \partial_{C'\otimes D} \circ (s' \otimes 1_D) + (s' \otimes 1_D) \circ \partial_{C'\otimes D}$$
$$1_{C\otimes D} - (g \otimes 1_D) \circ (f \otimes 1_D) = \partial_{C'\otimes D} \circ (s \otimes 1_D) + (s \otimes 1_D) \circ \partial_{C'\otimes D}$$

leaing to chain homotopy equivalence

$$C_\bullet \otimes D_\bullet \underset{g\otimes 1_D}{\overset{f\otimes 1_D}{\rightleftarrows}} C'_\bullet \otimes D_\bullet \ .$$

□

We would like to compare the following two functors

$$S_\bullet(- \times -), S_\bullet(-) \otimes S_\bullet(-) : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}_\bullet$$

which send

$$X \times Y \to S_\bullet(X \times Y) \quad \text{and} \quad S_\bullet(X) \otimes S_\bullet(Y).$$

We first observe that there exists a canonical isomorphism

$$\mathrm{H}_0(X \times Y) \simeq \mathrm{H}_0(X) \otimes \mathrm{H}_0(Y).$$

The Eilenberg-Zilber Theorem says that such initial condition determines a natural homotopy equivalent between the above two functors which are unique up to chain homotopy.

Theorem (Eilenberg-Zilber)

Then there exist natural transformations (Eilenberg-Zilber maps)

$$S_\bullet(- \times -) \underset{G}{\overset{F}{\rightleftarrows}} S_\bullet(-) \otimes S_\bullet(-)$$

which induce chain homotopy equivalence for every X, Y

$$S_\bullet(X \times Y) \underset{G}{\overset{F}{\rightleftarrows}} S_\bullet(X) \otimes S_\bullet(Y)$$

and the canonical isomorphism $\mathrm{H}_0(X \times Y) \simeq \mathrm{H}_0(X) \otimes \mathrm{H}_0(Y)$.

Such chain equivalence is unique up to chain homotopy. In particular, there are canonical isomorphisms

$$\mathrm{H}_n(X \times Y) = \mathrm{H}_n(S_\bullet(X) \otimes S_\bullet(Y)), \quad \forall n \geq 0.$$

Proof

Observe that any map $\Delta^p \stackrel{(\sigma_x,\sigma_y)}{\to} X \times Y$ factors through

$$\Delta^p \stackrel{\delta_p}{\to} \Delta^p \times \Delta^p \stackrel{\sigma_x \times \sigma_y}{\to} X \times Y$$

where $\Delta^p \stackrel{\delta_p}{\to} \Delta^p \times \Delta^p$ is the diagonal map. This implies that a natural transformation F of the functor $S_\bullet(- \times -)$ is determined by its value on $\{\delta_p\}_{p \geq 0}$. Explicitly

$$F((\sigma_x, \sigma_y)) = (\sigma_x \otimes \sigma_y)_* F(\delta_p).$$

Similarly, a natural transformation G of the functor $S_\bullet(-) \otimes S_\bullet(-)$ is determined by its value on $1_p \otimes 1_q$ where $1_p : \Delta^p \to \Delta^p$ is the identity map. Explicitly, for any $\sigma_x : \Delta^p \to X, \sigma_y : \Delta^q \to Y$,

$$G(\sigma_x \otimes \sigma_y) = (\sigma_x \times \sigma_y)_* G(1_p \otimes 1_q).$$

Therefore F and G are completely determined by

$$f_n := F(\delta_n) \in \bigoplus_{p+q=n} S_p(\Delta^n) \otimes S_q(\Delta^n)$$

$$g_n := \bigoplus_{p+q=n} G(1_p \otimes 1_q) \in \bigoplus_{p+q=n} S_n(\Delta^p \times \Delta^q).$$

We will use the same notations as in the discussion of Barycentric subdivision. Then

$$f_n \circ g_n \in S_n(\Delta^n \times \Delta^n), \quad g_n \circ f_n \in \bigoplus_{p+q=n} (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_n.$$

Let us denote the following chain complexes

$$C_n = \prod_{k\geq 0}(S_\bullet(\Delta^k)\otimes S_\bullet(\Delta^k))_{n+k}, \quad D_n = \prod_{m\geq 0}\left(\bigoplus_{p+q=m} S_{n+p+q}(\Delta^p\times\Delta^q)\right)$$

with boundary map

$$\partial+\tilde{\partial}: C_n\to C_{n-1}, \quad \partial+\tilde{\partial}: D_n\to D_{n-1}$$

as follows.

∂ is the usual boundary map of singular chain complexes

$$\begin{aligned} \partial: \quad & (S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k))_n \to (S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k))_{n-1} \\ \partial: \quad & S_n(\Delta^p \times \Delta^q) \to S_{n-1}(\Delta^p \times \Delta^q). \end{aligned}$$

$\tilde{\partial}$ is the map induced by composing with the face singular chain

$$\tilde{\partial} = \sum_k \partial \Delta^k \in \prod_k S_{k-1}(\Delta^k)$$

On C:

$$\tilde{\partial} : S_p(\Delta^{k-1}) \otimes S_q(\Delta^{k-1}) \to S_p(\Delta^k) \otimes S_q(\Delta^k)$$
$$\sigma_p \otimes \sigma_q \to \tilde{\partial} \circ \sigma_p \otimes \tilde{\partial} \circ \sigma_q$$

On D:

$$\tilde{\partial} : S_n(\Delta^p \times \Delta^q) \to S_n(\Delta^{p+1} \times \Delta^q) \oplus S_n(\Delta^p \times \Delta^{q+1})$$
$$\sigma_p \times \sigma_q \to (\tilde{\partial} \circ \sigma_p) \times \sigma_q + (-1)^{n-p} \sigma_p \times (\tilde{\partial} \circ \sigma_q).$$

Let $f = (f_n) \in C_0$ and $g = (g_n) \in D_0$. Then

$$F, G \text{ are chain maps} \Longleftrightarrow f, g \text{ are } 0\text{-cycles in } C_\bullet, D_\bullet$$

and natural chain homotopy of F, G are given by 0-boundaries.

We claim that

$$\mathrm{H}_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad \mathrm{H}_n(D_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

This implies that the initial condition completely determines chain maps F, G up to chain homotopy.

We sketch a proof here. There exists a spectral sequence with

$$E_1\text{-page} : \mathrm{H}(-, \partial)$$
$$E_2\text{-page} : \mathrm{H}(\mathrm{H}(-, \partial), \tilde{\partial})$$

and converging to $\partial + \tilde{\partial}$-homology. We need to use a stronger version of convergence than we have discussed before, which is guaranteed by the choice of direct product (so formal series is convergent) instead of direct sum in the definition of C_n and D_n.

For $C_\bullet$, the E_1-page $\mathrm{H}_\bullet(C_\bullet, \partial)$ is

$$\mathrm{H}_n(C_\bullet, \partial) = \prod_{k\geq 0} \mathrm{H}_n(S_\bullet(\Delta^k) \otimes S_\bullet(\Delta^k)) = \begin{cases} \prod_{k\geq 0} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

$\tilde{\partial}$ acts on this E_1-page as

$$\tilde{\partial} : \prod_{k\geq 0} \mathbb{Z} \to \prod_{k\geq 0} \mathbb{Z} \qquad (n_k)_{k\geq 0} \to (m_k)_{k\geq 0}$$

$$\text{where} \quad m_k = \frac{1}{2}(1+(-1)^k)n_{k-1}.$$

In components, this can be represented by

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \cdots$$

The $\tilde{\partial}$-homology is now $\mathbb{Z}$ concentrated at degree 0. It follows that $E_2 = E_3 = \cdots = E_\infty$ and therefore

$$\mathrm{H}_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

The computation in the case of $D_\bullet$ is similar.

Let us now analyze the composition $F \circ G$ and $G \circ F$. We similarly form the chain complexes

$$C'_n = \prod_{k \geq 0} S_{n+k}(\Delta^k \times \Delta^k), \quad D'_n := \prod_{m \geq 0} \bigoplus_{p+q=m} (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_{n+p+q}$$

with boundary map $\partial + \tilde{\partial}$ defined similarly.

Homology of $C'_\bullet$ controls natural chain maps of $S_\bullet(X \times Y)$ to itself up to chain homotopy, and similarly for $D'_\bullet$.

We still have

$$\mathrm{H}_n(C'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}, \quad \mathrm{H}_n(D'_\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}.$$

It follows that $F \circ G$ and $G \circ F$ are both naturally chain homotopic to the identity map. The theorem follows. □

An explicit construction of G can be described as follows: given $\sigma_p : \Delta^p \to X, \sigma_q : \Delta^q \to Y$,

$$G(\sigma_p \otimes \sigma_q) : \Delta^p \times \Delta^q \to X \times Y$$

where we have to chop $\Delta^p \times \Delta^q$ into $p+q$-simplexes. This is the shuffle product.

An explicit construction of F can be given by the Alexander-Whitney map described as follows.

Definition

Given a singular n-simplex $\sigma : \Delta^n \to X$ and $0 \leq p, q \leq n$, we define

- the front p-face of σ to be the singular p-simplex

$$_p\sigma : \Delta^p \to X, \quad {}_p\sigma(t_0, \cdots, t_p) := \sigma(t_0, \cdots, t_p, 0, \cdots, 0)$$

- the back q-face of σ to be the singular q-simplex

$$\sigma_q : \Delta^q \to X, \quad \sigma_q(t_0, \cdots, t_q) := \sigma(0, \cdots, 0, t_0, \cdots, t_q).$$

Definition

Let X, Y be topological spaces. Let

$$\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$$

be the projections. We define the Alexander-Whitney map

$$AW : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$$

by the natural transformation given by the formula

$$AW(\sigma) := \sum_{p+q=n} {}_p(\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma)_q.$$

Theorem

The Alexander-Whitney map is a chain homotopy equivalence.

Proof.

It can be checked that AW is a natural chain map which induces the canonical isomorphism

$$\mathrm{H}_0(X \times Y) \to \mathrm{H}_0(X) \otimes \mathrm{H}_0(Y).$$

So AW is a chain homotopy equivalence by Eilenberg-Zilber Theorem. □

Künneth formula

Theorem (Algebraic Künneth formula)

Let $C_\bullet$ and $D_\bullet$ be chain complex of free abelian groups. Then there is a split exact sequence

$$0 \to (\mathrm{H}_\bullet(C) \otimes \mathrm{H}_\bullet(D))_n \to \mathrm{H}_n(C_\bullet \otimes D_\bullet) \to \mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_{n-1} \to 0.$$

Here

$$\mathrm{Tor}(\mathrm{H}_\bullet(C), \mathrm{H}_\bullet(D))_k = \bigoplus_{p+q=k} Tor(\mathrm{H}_p(C), \mathrm{H}_q(D)).$$

Proof

Using the freeness of $C_\bullet$ we can show that

$$\mathrm{H}_\bullet(C_\bullet \otimes D_\bullet) = \mathrm{H}_\bullet(C_\bullet \otimes \mathrm{H}_\bullet(D)).$$

Applying Universal Coefficient Theorem for Homology, we find

$$0 \to \mathrm{H}_p(C)\otimes\mathrm{H}_q(D) \to \mathrm{H}_{p+q}(C_{\bullet-q}\otimes\mathrm{H}_q(D)) \to \mathrm{Tor}(\mathrm{H}_{p-1}(C), \mathrm{H}_q(D)) \to 0$$

Summing over p, q gives the theorem. □

Theorem (Künneth formula)

For any topological spaces X, Y and $n \geq 0$, there is a split exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \to \mathrm{H}_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0.$$

Proof.

This follows from the Eilenberg-Zilber Theorem and the algebraic Künneth formula.

□